On A Cauchy Problem In A Hilbert Space With Operator Coefficients

Gafarov Ilg'or Ahmedjanovich¹ and Eshmatov Davron Abduvaxobovich²

¹Namangan Engineering Construction Institute, Namangan, Republic of Uzbekistan
²Namangan Engineering Construction Institute, Namangan, Republic of Uzbekistan

Abstract – This article provides feedback on the operator coefficients of the Cauchy problem in the Hilbert phase.

Keywords – Function Of Scalar Argument, Generating Vector, Inverse Differentiation.

Consider the following Cauchy problem in Hilbert space $H$

$$\frac{d^n u(t)}{dt^n} + \sum_{k=1}^{n} A_k(t) \frac{d^{n-k} u(t)}{dt^{n-k}} = f(t),$$  \hspace{1cm} (1)

$$\frac{d^k u(t)}{dt^k} \bigg|_{t=0} = u^{(k)}(0), \hspace{1cm} k = 0, 1, \ldots, k - 1,$$  \hspace{1cm} (2)

Where $u(t), f(t)$ – scalar argument functions $t \in [0, T]$ with values in $D(A) = \cap D(A_k(t)) \subset H$, $A_k(t)$ – generally speaking, linear unbounded operators acting in $H$. As is known [1], such problems are usually ill-posed in the classical sense. Therefore, we will investigate for conditional correctness.

Previously, such a problem for “$l$-correctness” was considered in [2]. In this work, the conditional stability is estimated in another way.

Let be $H$ - separable Hilbert space. Let us recall some concepts that we will use in what follows.

The decomposition of unity is a one-parameter family of projecting operators $E_{s}$, given in a finite or infinite interval $[\alpha, \beta]$ and satisfying the following conditions:

a) $E_\lambda E_\mu = E_s \left( S = \min \{\lambda, \mu\} \right)$;

b) in the sense of strong convergence $E_{\lambda - 0} = E_\lambda \left( \alpha < \lambda < \beta \right)$;

c) $E_\alpha = 0$, $E_\beta = I$.

Having an interval $\Delta = [\lambda', \lambda] \subset [\alpha, \beta]$, we will be the difference $E_{\lambda'} - E_{\lambda}$ denote $E_{\Delta}$. 

Corresponding Author: Gafarov Ilg'or Ahmedjanovich
The spectrum of a self-adjoint operator is called simple if there is such a vector \( g \in H \) (generating vector), which is dense in the linear span of a set of vectors \( E(\Delta)g \), where \( \Delta \) runs through the set of all intervals on the number axis.

Let the operators \( A_k(t) \) in equation (1) have strongly continuous derivatives up to the order \( n - k \) (\( k = 1, 2, \ldots, n \)) inclusive. To equation (1) we apply \( n \) times the inverse differentiation operator

\[
u(t) + \frac{1}{(n-1)!} \left\{ \int_0^t (t-\tau)^{n-1} A_1(\tau) u^{(n-1)}(\tau) d\tau + \ldots + \int_0^t (t-\tau)^{n-1} A_n(\tau) u(\tau) d\tau \right\} =
\]

\[
= \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau + \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0).
\]

In each integral of the last equation, the derivatives \( u^{(k)}(t) \) we transfer to other factors. Then we get

\[
u(t) + (Nu)(t) = \varphi(t), \quad (3)
\]

Where

\[
(Nu)(t) = \int_0^t N_1(t,\tau) u(\tau) d\tau,
\]

\[
N_1(t,\tau) = \frac{1}{(n-1)!} \sum_{m=0}^{n-1} \sum_{k=0}^{m} (-1)^{k+m-1} \frac{C^k}{(n-1)!} (t-\tau)^{n-k-1} A^{(m-k)}_{n-m}(\tau),
\]

\[
\varphi(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau + \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) +
\]

\[
+ \sum_{l=1}^{n-1} \sum_{k=1}^{n-l} (-1)^{n-l} \frac{t^{n+m-k-1}}{(n+m-k-1)!} A^{(m-l)}_{n-l}(0) u^{(n-l-k)}(0).
\]

Suppose now that there is a constant self-adjoint completely continuous operator with simple spectrum such that

\[
\|S\| < 1, \quad \|SN_i\| \leq M, \quad \|SN\| = \|NS\| \leq 1. \quad (4)
\]

By condition \( S = \int f(\lambda) dE_\lambda \) and \( \nu(t) = \int u(\lambda,t) dE_\lambda g \) in \( D(S) \), where \( g \) — generating vector.

We will assume that \( |f(\lambda)| \geq \lambda^{-1} \), then the following holds.

Theorem. Let be \( \|\varphi(t)\| \leq \varepsilon \) and for some \( \alpha, c > 0 \)

\[
|u(\lambda,t)| \leq c \cdot \lambda^{-\alpha}.
\]

Then

\[
\|u(t)\| \leq 2c^{k_\varepsilon} \cdot (2k_\varepsilon \cdot \varepsilon)^{k_\varepsilon},
\]

where \( k_\varepsilon \) equation solution \( (k!)^{-1} = k \cdot \varepsilon \).

Evidence. From equation (3) after applying the operator \( S \), we get

\[
Su(t) = A(t) \cdot u(t) + S \varphi(t),
\]

(5)
Where \( A(t) = N(t) \cdot S \).

We apply to equation (5) the operator \( S \):

\[
S^2 u(t) = S A(t) u(t) + S^2 \varphi(t).
\]

Hence, taking into account (5)

\[
S^2 u(t) = A(t) S u(t) + S^2 \varphi(t) = A^2(t) u(t) + [A(t) S + S^2] \varphi(t).
\]

To the last equality we apply again the operator \( S \)

\[
S^3 u(t) = A^2(t) S u(t) + [A(t) S^2 + S^3] \varphi(t) =
A^3(t) u(t) + [A^2(t) S + A(t) S^2 + S^3] \varphi(t).
\]

By doing the same \( k \) once we arrive at the following equation

\[
S^k u(t) = A^k(t) u(t) + \left[ \sum_{i=1}^{k} A^{k-i}(t) S^i \right] \varphi(t).
\]

Hence, using (4), we obtain

\[
\|S^k u(t)\| \leq \|A^k(t)\| \cdot \|u(t)\| + k \cdot \varepsilon \quad (6)
\]

Applying the Cauchy formula, taking into account (4), it is easy to verify that

\[
\|A^k(t)\| \leq (Mt)^k \cdot (k!)^{-1} \leq (k!)^{-1}.
\]

Now if \( c \) is such that \( \|u(t)\| \leq 1 \), from (6) it follows

\[
\|S^k u(t)\| \leq 2k \varepsilon \quad (7)
\]

Hence we conclude for a fixed \( \lambda_0 \)

\[
\int_{|\lambda| < \lambda_0} |u(\lambda, t)|^2 \left( dE_{g} g, g \right) \leq 4k^2 \varepsilon^2 \cdot \lambda_0^{2k} \quad (8)
\]

Really,

\[
\lambda_0^{-2k} \int_{|\lambda| < \lambda_0} |u(\lambda, t)|^2 \left( dE_{g} g, g \right) \leq \int_{|\lambda| < \lambda_0} |\lambda|^{-2k} |u(\lambda, t)|^2 \left( dE_{g} g, g \right) \leq
\leq \int \lambda^{-2k} |u(\lambda, t)|^2 \left( dE_{g} g, g \right) \leq \int f(\lambda)^{2k} |u(\lambda, t)|^2 \left( dE_{g} g, g \right) \leq 4k^2 \varepsilon^2.
\]

On the other hand

\[
\int_{|\lambda| > \lambda_0} |u(\lambda, t)|^2 \left( dE_{g} g, g \right) \leq c^2 \int_{|\lambda| > \lambda_0} |\lambda|^{-2a} \left( dE_{g} g, g \right) \leq c^2 \cdot |\lambda_0|^{-2a} \int \left( dE_{g} g, g \right) \leq c^2 \lambda_0^{-2a}.
\]

From here and from inequality (8) we obtain:

\[
\|u(t)\|^2 = \int |u(\lambda, t)|^2 \left( dE_{g} g, g \right) = \left( \int_{|\lambda| < \lambda_0} + \int_{|\lambda| > \lambda_0} \right) |u(\lambda, t)|^2 \left( dE_{g} g, g \right) \leq 4k^2 \varepsilon^2 \lambda_0^{2k} + c^2 \lambda_0^{-2a}.
\]

Finding the optimal \( \lambda_0 \) (equating the terms on the right-hand side of the last inequality) we obtain the statement of the theorem.

Comment. It should be noted that from the equation \( (k!)^{-1} = k \cdot \varepsilon \) follows that \( k \cdot \varepsilon \rightarrow \infty \).
at \( \varepsilon \rightarrow 0 \). Hence, it is easy to see that

\[
\frac{k_{\varepsilon}}{C_{k_{\varepsilon}+\alpha}} \cdot \left(2k_{\varepsilon} \cdot \varepsilon \right)^{\frac{\alpha}{k_{\varepsilon}+\alpha}} \rightarrow 0 \quad \text{at} \quad \varepsilon \rightarrow 0. (*)
\]

REFERENCES
